

Revisiting Impossible Quantum Operations Using Principles of No-Signaling and Nonincrease of Entanglement Under LOCC

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Abstract In this letter, we show the impossibility of the general operation introduced by Pati (Phys. Rev. A 66:062319, 2002) using two different but consistent principles (i) no-signaling (ii) non increase of entanglement under LOCC.

The rules of quantum mechanics makes certain processes impossible. Neither we can clone an quantum state [1] nor we can delete one of the two identical copies of an arbitrary quantum state [2]. Operations like cloning, conjugation, complementing etc comes under the broad heading ‘General Impossible Operations’ [3]. In the recent past impossibility of operations like cloning, flipping and deletion had been shown from two fundamental principles namely (i) principle of no-signaling and (ii) nonincrease of entanglement under LOCC [4–6].

In this letter we show that ‘General Impossible Operations’ [3] which will act on the tensor product of an unknown quantum state and blank state at the input port to produce the original state along with a function of the original state at the output port is not feasible in the quantum world from two basic principles: No signaling principle and Thermodynamical law of Entanglement. The letter is different from Ref. [3] in the sense that here we used two basic principles namely (i) principle of no-signaling and (ii) thermodynamic law of entanglement to prove the impossibility, instead of unitarity of quantum mechanics in [3].

Suppose there is a singlet state consisting of two particles shared by two distant parties Alice and Bob. The state is given by,

$$|\chi\rangle_{12} = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|0\rangle) = \frac{1}{\sqrt{2}}(|\psi\rangle|\bar{\psi}\rangle - |\bar{\psi}\rangle|\psi\rangle) \quad (1)$$

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where $\{|\psi\rangle, |\bar{\psi}\rangle\}$ are mutually orthogonal spin states or in other words they are mutually orthogonal polarizations in case of photon particles. Alice is in possession of the first particle and Bob is in possession of the second particle.

No-signaling principle states that if one distant partner (say, Alice) measures her particle in any one of the two basis namely $\{|0\rangle, |1\rangle\}$ and $\{|\psi\rangle, |\bar{\psi}\rangle\}$, then measurement outcome of the other party (say, Bob) will remain invariant. At this point one might ask an interesting question: Is there any possibility for Bob to know the basis in which Alice measures her qubit, if he apply the operations defined as ‘General Impossible operation’ [1] on his qubit.

Let us consider a situation where Bob is in possession of a hypothetical machine whose action in two different basis $\{|0\rangle, |1\rangle\}$ and $\{|\psi\rangle, |\bar{\psi}\rangle\}$ is defined by the transformation,

$$|i\rangle|\Sigma\rangle \longrightarrow |i\rangle|F(i)\rangle \quad (i = 0, 1), \tag{2}$$

$$|j\rangle|\Sigma\rangle \longrightarrow |j\rangle|F(j)\rangle \quad (j = \psi, \bar{\psi}) \tag{3}$$

where $\{|\Sigma\rangle\}$ is the ancilla state attached by Bob. These set of transformations was first introduced by Pati in [1].

After the application of the transformation defined in (2–3) by Bob on his particle, the singlet state takes the form

$$\begin{aligned} |\chi\rangle|\Sigma\rangle &\rightarrow \frac{1}{\sqrt{2}}(|0\rangle|1\rangle|F(1)\rangle - |1\rangle|0\rangle|F(0)\rangle) \\ &= \frac{1}{\sqrt{2}}(|\psi\rangle|\bar{\psi}\rangle|F(\bar{\psi})\rangle - |\bar{\psi}\rangle|\psi\rangle|F(\psi)\rangle). \end{aligned} \tag{4}$$

Now Alice can measure her particle in two different basis. If Alice measures her particle in $\{|0\rangle, |1\rangle\}$, then the reduced density matrix describing Bob’s subsystem is given by,

$$\rho_B = \frac{1}{2}[|1\rangle\langle 1| \otimes |F(1)\rangle\langle F(1)| + |0\rangle\langle 0| \otimes |F(0)\rangle\langle F(0)|]. \tag{5}$$

On the other hand if Alice measures her particle in the basis $\{|\psi\rangle, |\bar{\psi}\rangle\}$ then the state described by the reduced density matrix in the Bob’s side is given by,

$$\rho_B = \frac{1}{2}[|\bar{\psi}\rangle\langle \bar{\psi}| \otimes |F(\bar{\psi})\rangle\langle F(\bar{\psi})| + |\psi\rangle\langle \psi| \otimes |F(\psi)\rangle\langle F(\psi)|]. \tag{6}$$

Since the statistical mixture in (5) and (6) are different, so this would have allow Bob to distinguish the basis in which Alice has performed the measurement and this lead to superluminal signaling. But this is not possible from the principle of ‘no-signaling’, so we arrive at a contradiction. Hence, we conclude from the principle of no-signaling that the transformation defined in (2–3) is not possible in the quantum world.

Now we will show that the operations referred to as ‘General impossible operations’ is not feasible in the quantum world from the principle of non increase of entanglement under LOCC [in general one can only claim that it is conserved under a bilocal unitary operation].

Let Alice and Bob share an entangled state

$$|\Psi\rangle_{AB} = \frac{1}{\sqrt{2}}[|0\rangle|\psi_1\rangle + |1\rangle|\psi_2\rangle]|\Sigma\rangle. \tag{7}$$

If Bob operates ‘General Impossible operations’ on his qubit, then the entangled state takes the form

$$|\Psi\rangle_{AB}^I = \frac{1}{\sqrt{2}}[|0\rangle|\psi_1\rangle|F(\psi_1)\rangle + |1\rangle|\psi_2\rangle|F(\psi_2)\rangle]. \tag{8}$$

It can be easily shown that the reduced density matrix describing Alice’s subsystem before and after the physical operation will be different. The reduced density matrices before and after the application of general operations are given by,

$$\rho_A = \frac{1}{2}[|0\rangle\langle 0| + |1\rangle\langle 1| + |0\rangle\langle 1|(\langle\psi_2|\psi_1\rangle) + |1\rangle\langle 0|(\langle\psi_1|\psi_2\rangle)] \tag{9}$$

and

$$\rho_A^I = \frac{1}{2}[|0\rangle\langle 0| + |1\rangle\langle 1| + |0\rangle\langle 1|(\langle\psi_2|\psi_1\rangle)(\langle F(\psi_2)|F(\psi_1)\rangle) + |1\rangle\langle 0|(\langle\psi_1|\psi_2\rangle)(\langle F(\psi_1)|F(\psi_2)\rangle)]. \tag{10}$$

The largest eigenvalues corresponding to these density matrices are given by, $\lambda = \frac{1}{2} + \frac{|\alpha|}{2}$ and $\lambda^I = \frac{1}{2} + \frac{\sqrt{\alpha^2\alpha'^2}}{2}$ respectively, where $\alpha = \langle\psi_1|\psi_2\rangle = \langle F(\psi_1)|F(\psi_2)\rangle$ and $|\psi_1\rangle = a|0\rangle + b|1\rangle$, $|\psi_2\rangle = c|0\rangle + d \exp(i\theta)|1\rangle$.

Let us consider $\alpha = X + iY$ where $X = ac + bd \cos \theta$ and $Y = bd \sin \theta$.

From the principle of nonincrease of entanglement we can write, $E(|\psi\rangle_{AB}^I) \leq E(|\psi\rangle_{AB})$, which implies $\lambda \leq \lambda^I$. That is, we are assuming that there is no increase of entanglement.

Therefore $\lambda \leq \lambda^I \Rightarrow |\alpha|^2 \geq 1 \Rightarrow X^2 + Y^2 \geq 1 \Rightarrow \cos \theta \geq \frac{1-(a^2c^2+b^2d^2)}{2abcd}$.

Now we observe that $(a - c)^2 + (b - d)^2 \geq 0 \Rightarrow ac + bd \leq 1$. Again squaring both sides, the inequality reduces to $a^2c^2 + b^2d^2 + 2abcd \leq 1 \Rightarrow \frac{1-(a^2c^2+b^2d^2)}{2abcd} \geq 1 \Rightarrow \cos \theta \geq 1$. The equality holds when $a = c$ and $b = d$. So we arrive at a contradiction when $a \neq c$ and $b \neq d$. Hence $\lambda > \lambda^I$. This violates the principle of no increase of entanglement under LOCC. This proves the impossibility of ‘General Impossible Operations’ from the principle of nonincrease of entanglement under LOCC.

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